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FORMULAS FOR CALCULATING DEFORMATIONS AND NATURAL FREQUENCY OF FREE VIBRATIONS OF A HEXAGONAL TOWER*

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Statement of the problem. A scheme of a six-sided prismatic statically determinate spatial girder is proposed. The task is set by induction to derive formulas for the dependence of the deflection of the structure and the lower limit of the main frequency of natural vibrations on the number of panels along the height of the prism.
Materials and methods. The forces in the rods along with the reactions of the supports are found in an analytical form by means of the method of cutting nodes in the Maple symbolic mathematics system. One of the nodes at the base of the truss has a spherical support, one has a cylindrical support, the remaining four supports are racks. The top deflection is determined by the Maxwell-Mohr formula. From the analysis of the sequences of coefficients in the formulas for individual structures with a different number of panels, their common members are determined, which are included in the desired calculation formula. The Dunkerley method is used to estimate the first frequency of free oscillations.
Results. For various types of loads, formulas for the dependence of girderf deflections on the number of panels are obtained. The coefficients in the solution are polynomial in the number of panels. The derived analytical dependence of the first frequency on the number of panels in comparison with the numerical solution has a small error, which decreases with increasing number of panels.
Conclusions. A design of an axisymmetric statically determinate tower-type girder has been developed, which allows analytical solutions to the problem of deflection and the problem of the first natural frequency for an arbitrary number of panels. The resulting formulas can be used to assess the accuracy of numerical solutions and for preliminary calculations of models of structures of this type.

Keywords: hexagonal prism, spatial girder, deflection, induction, Maple, natural frequency, Dunkerley method, analytical solution.

Introduction. To calculate deformations and natural frequencies of spatial building structures in engineering practice, numerical methods based on the finite element method are used [2, 6, 18]. In calculations of complex systems containing a large number of elements, an inevitable error occurs in the accumulation of rounding errors resulting in a a loss of accuracy [2, 21]. If the structure being calculated is a regular one, inductive methods can be applied to it for obtaining analytical dependencies of the value under study on the order of the system, e.g., on the number of panels or on occasionally re-

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peating groups of rods. Using this method in the computer mathematics system *Maple* [23, 24], some solutions to the problem of deflection were obtained for flat lattice [9, 11, 15], arched [10, 20] and spatial girders[14, 17]. Formulas for the lower limit of the first natural frequency using the Donckerley method for an arbitrary number of panels were obtained in [16, 19].

Analytical methods using variational principles in the system of symbolic mathematics were used in [3—5]. The variational method is applicable for calculating regular and arbitrary structures. The induction method for calculating structures is used for regular systems with occasionally repeating groups of rods (e.g., panels) and typically gives simple calculation formulas [1]. The method is based on a generalization of a series of relatively simple analytical solutions to the problem of deflection of girders to an arbitrary order of their regularity. The induction method is used mainly for statically determinate systems. The problem of the existence of statically definable regular constructions was first addressed by R. G. Hutchinson, N. A. Fleck, F. W. Zok, R. M. Latture, M. R. Begley [7, 8, 22]. Schemes of flat statically definable regular trusses and formulas for calculating their deflections in the case of various loads are provided in the author's reference books [12, 13].

In this study, we set forth a diagram of a statically determinate spatial tower-type truss (Fig. 1, 2), consisting of six identical faces with a cross-shaped lattice. The farm has n tiers of height h, except for the upper dome-shaped one of height $h/2$. The supports of the structure are four posts, a spherical hinge support A and a cylindrical one B.

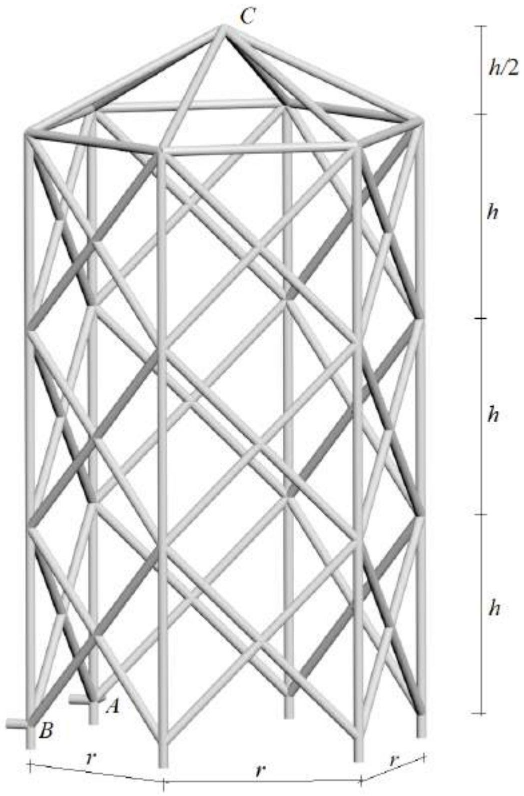

Fig. 1. Girder, $n = 4$

The braces of the cross-shaped lattice have the length $d=\sqrt{a^2+r^2}$. The hexagonal dome of height $h/2$ rests on the nodes of the upper contour. The racks of the tiers have a height of h. The girder contains $n_s=18n+3$ rods including nine support ones. The task is to derive analytical dependences of the deflection of a structure under the influence of various loads on the number of panels and to obtain a formula for a lower estimate of the first frequency of its natural vibrations.

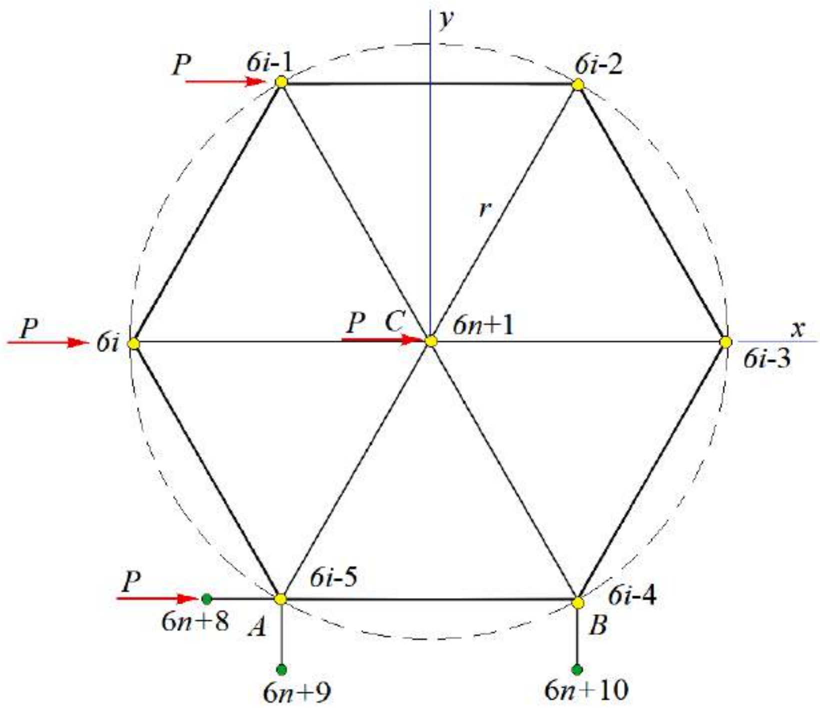

Fig. 2. Numbering of nodes of the *i*-th circuit

1. Calculation of the forces. We will calculate the forces in the rods and all the transformations necessary to derive the required formulas in the *Maple* system of symbolic mathematics [1, 24].

The coordinates of the nodes are entered into the software program. The origin of coordinates is chosen on the axis of a cylinder of radius r circumscribed around the tower (Fig. 2). The coordinates of the lattice nodes have the form:

$$x_{i+6(j-1)}=r\cos\varphi, y_{i+6(j-1)}=r\sin\varphi, z_{i+6(j-1)+1}=h(j-1), \varphi=\pi i/3,\ i=1,..,6,\ j=1,..,n.$$

The coordinates of the hinges of the racks on the base are

$$x_{i+6n+1}=r\cos\varphi, y_{i+6n+1}=r\sin\varphi, z_{i+6n+1}=-u, \varphi=\pi i/3,\ i=1,..,6,$$

where u is an arbitrary number, e.g., 1 (length of support posts). The racks are assumed to be rigid, and their lengths are not included in the solution. Vertex C coordinates are

$$x_{6n+1}=0, y_{6n+1}=0, z_{6n+1}=h(n-1/2).$$

The structure of rod connections is specified by the lists T_i, $i=1,...,n_s$ of the numbers of the ends of the corresponding rods. The faces (vertical rods) are coded, e.g., in the following way:

$$T_{6n+6(j-1)}=[6j,6j+1],$$
$$T_{i+6(j-1)}=[i+6(j-1),i+6j+1],\ i=1,..,5,\ j=1,..,n-1.$$

Based on the data on the coordinates of the nodes and the order of connecting the rods into nodes, the coefficients of the equilibrium equations of the nodes are calculated in projections onto three coordinate axes. The system of linear equilibrium equations has the form: $\mathbf{AS}=\mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients (direction cosines of the forces in the rods), $\mathbf{S}$ is the vector of all the unknown forces. The number of the forces also includes nine support reactions. The vector components $\mathbf{B}$ are node forces. For node *i* three consecutive elements of the vector are highlighted $\mathbf{B}$. The elements with type numbers 3*i* - 2 include external horizontal forces in projection on the *x* axis, and elements 3*i* - 1 - on the *y* axis. The vertical loads are in elements numbered *3i*.

Under the action of a uniform vertical (Fig. 3) load on the six upper nodes and the vertex, the non-zero elements of the vector elements numbered such as $3i-2$ external horizontal forces are entered in the projection on the x axis, in elements 3*i* - 1 - on the y axis. The vertical loads are in elements numbered 3*i*.

Under the action of a uniform vertical (Fig. 3) load on the six upper nodes and the vertex, the non-zero elements of the vector have the form: $B_{3i}=-G,\ i=6n-5,...,6n+1$.

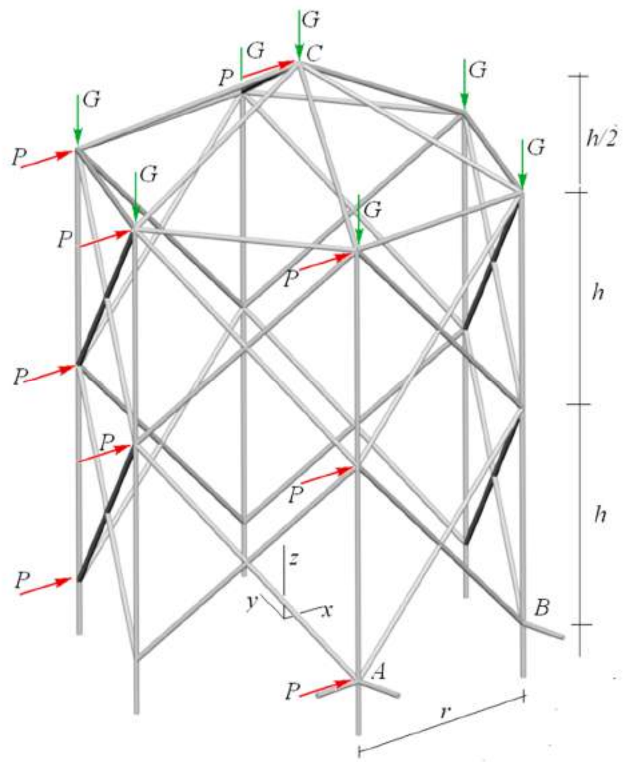

Fig. 3. Lateral (wind) load *P* and a vertical one *G* onto the girder, *n* = 3

2. Deflection. Vertical load. To calculate the vertical displacement of the top C of the structure, the Maxwell-Mohr formula is used: $\Delta = \sum_{j=1}^{n_s-9} \frac{S_j s_j l_j}{EF}$, where E is the elasticity modulus of the rods; F is their cross-sectional area; l_j and S_j is the length and force in the j-th rod under the load; s_j is the force of a vertical force. Nine support rods are assumed to be non-deformable and are not included in the total.

For different n, the following expressions for deflection are obtained:

$$\begin{aligned}
&\Delta_1 = G(c^3 + 8r^3)/(12h^2 EF),\\
&\Delta_2 = G(c^3 + 14h^3 + 8r^3)/(12h^2 EF),\\
&\Delta_3 = G(c^3 + 28h^3 + 8r^3)/(12h^2 EF),\\
&\Delta_4 = G(c^3 + 42h^3 + 8r^3)/(12h^2 EF),...,
\end{aligned}$$

where $c = \sqrt{4r^2 + h^2}$. The coefficients at c^3 and r^3 do not change. A common member of the sequence of the coefficients at h^3 can be found by using *Maple* tools. As a result, the dependence of the deflection on the number of panels will take the form:

$$\Delta = G\left(C_1 r^3 + C_2 c^3 + C_3 h^3\right)/\left(h^2 EF\right),$$

where $C_1 = 2/3,\ C_2 = 1/12,\ C_3 = 7(n-1)/6.$

3. Horizontal load. The wind load in the x-axis direction can be modeled by nodal forces on the three leeward faces of the truss (Fig. 3):

$$\begin{aligned}
&B_{3i-2} = P,\ i = 6n+1,\\
&B_{3j-2} = P,\ j = 6i, 6i-1, 6i-5,\ i = 1,...,n-1.
\end{aligned}$$

The deflection is calculated using the horizontal displacement of vertex C. The vector of the right side of the system of equilibrium equations for nodes in projection onto the coordinate axes for identifyng the forces s_j in the Maxwell-Mohr formula has a single non-zero component: $B_{3i-2} = 1,\ i = 6n+1$. The sequence of solutions turns out to be more complex:

$$\begin{aligned}
&\Delta_1 = P(c^3 + r^3)/(4a^2 EF),\\
&\Delta_2 = P(21c^3 + 396d^3 + 652h^3 + 164r^3)/(24a^2 EF),\\
&\Delta_3 = P(18c^3 + 726d^3 + 5112h^3 + 19r^3)/(12a^2 EF),\\
&\Delta_4 = P(17c^3 + 1040d^3 + 15832h^3 + 140r^3)/(8a^2 EF),\\
&\Delta_5 = P(33c^3 + 2724d^3 + 71928h^3 + 35r^3)/(24a^2 EF),\dots .
\end{aligned}$$

In the general case, we have the form of the solution:

$$\Delta = P\left(C_1 r^3 + C_2 c^3 + C_3 d^3 + C_4 h^3\right)/\left(r^2 EF\right). \tag{1}$$

To determine the coefficients in (1), which are common members of the sequence of coefficients at powers of sizes, the operators rgf_findrecur and rsolve of the *Maple* system are used. The coefficients in (1) have the form:

$$
\begin{aligned}
&C_1=(8(7(-1)^n+9)n-41(-1)^n-51)/24,\\
&C_2=(5n-3)/8,\ C_3=(53n^2-91n-(-1)^n+37)/4,\\
&C_4=(950n^4-3076n^3+3436n^2-4(14(-1)^n+397)n+17(-1)^n+239)/48.
\end{aligned}
\tag{2}
$$

4. Numerical example. To construct graphs for solving the problem of wind load (1), (2), the height of the structure is fixed $H=nh$ and the total load on all nodes of the side faces $P_{sum}=P(3n+1)$. A dimensionless deflection of the top C is introduced: $\Delta'=\Delta EF/(P_{sum}H)$. The obtained dependence of the deflection on the radius reveals a minimum (Fig. 4). That enables one to choose the optimal ratio of the radius r and height H of the truss in terms of rigidity. As the height h increases, so does the value of the optimal radius r.

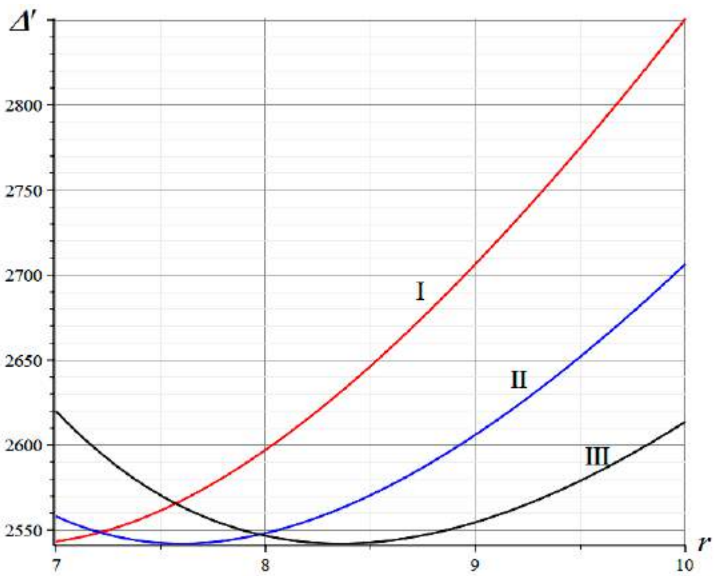

Fig. 4. Dependence of the deflection on the radius r with a horizontal load: n = 4;
I — $H=9$ m;
II — $H=10$ m;
III — $H=11$ m

Deflection curves have asymptotes whose angle of inclination can be calculated using the limit operator of the *Maple* system. The angle depends on the number of panels:

$$\lim_{r\to\infty}\Delta'/r=(954n^3+24(7(-1)^n-31)n^2-(85(-1)^n+57)n-47(-1)^n+99)/(24H).$$

5. Estimation of the first natural frequency of oscillations of the girder. Let us consider a model of horizontal vibrations of a truss with masses m located in all its nodes, except for the six support ones. The number of degrees of freedom in this formula is $N=6n-5$. The calculation of the oscillation frequencies of a system with a lot of degrees of freedom is possible only in numerical form. A lower estimate of the first frequency can be obtained analytically using the Donckerley method:

$$\omega_D^{-2} = \sum_{k=1}^{N} \omega_k^{-2}, \tag{3}$$

where ω_k is the oscillation frequency of a mass m in the node of the girder. The vibration equation written for one mass has a scalar form:

$$m\ddot{x}_k + d_k x_k = 0,$$

where x_k is the horizontal displacement of a mass; $\ddot{x}_k$ is the acceleration; d_k is the rigidity coefficient. The oscillation frequency (partial frequency) of the mass is $\omega_k = \sqrt{d_k / m}$. In order to identify the rigidity coefficient, the inverse of the compliance coefficient, the Maxwell-Mohr formula is used:

$$\delta_k = 1/d_k = \sum_{\alpha=1}^{n_s-9} \left(\tilde{S}_\alpha^{(k)}\right)^2 l_\alpha / (EF).$$

Let us denote $\tilde{S}_\alpha^{(k)}$ are the forces in the rod with the number from the action of a unit horizontal force applied to node k where the mass is located. According to (3):

$$\omega_D^{-2} = m\sum_{k=1}^{N} \delta_k = m\Delta_n. \tag{4}$$

Sequentially calculating the sum Δ_n, a general view of the solution is found:

$$\Delta_n = (C_{1,n}r^3 + C_{2,n}c^3 + C_{3,n}d^3 + C_{4,n}h^3)/(r^2 EF), \tag{5}$$

and the following sequence of the formulas is obtained:

$$\begin{aligned}
\Delta_2 &= 8(r^3 + d^3 + h^3)/(3r^2 EF),\\
\Delta_3 &= (124r^3 + 23c^3 + 1110d^3 + 6620h^3)/(9r^2 EF),\\
\Delta_4 &= 2(217r^3 + 23c^3 + 1599d^3 + 20778h^3)/(9r^2 EF),\\
\Delta_5 &= (110r^3 + 23c^3 + 2116d^3 + 47740h^3)/(3r^2 EF),\dots.
\end{aligned}$$

For sequence 8/3, 6620/9, 13852/3, 47740/3…. of the coefficients at h^3 the operator rgf_findrecur yields a recurrent solution:

$$C_{4,n} = 3C_{4,n-1} - C_{4,n-2} - 5C_{4,n-3} + 5C_{4,n-4} + C_{4,n-5} - 3C_{4,n-6} + C_{4,n-7}.$$

In order to obtain this equation, it was necessary to write the analytical solutions for girders with the number of panels from 2 to 14. The rsolve operator yields a solution for the coefficient $C_{4,n}$

$$C_{4,n} = (2676n^4 - 12128n^3 + 18934n^2 - 102n(-1)^n - 11974n + 165(-1)^n + 2555)/36. \tag{6}$$

Similarly, but in a somewhat simpler manner, we obtain the remaining coefficients of the desired formula:

$$C_{1,n} = (102n(-1)^n + 308n - 201(-1)^n - 571)/18,$$
$$C_{2,n} = 23(n-2)/9, \qquad (7)$$
$$C_{3,n} = (344n^2 - 1006n - 5(-1)^n + 657)/6.$$

We will estimate the error of the lower approximation of solution (4—7) by comparing it with the lowest spectrum frequency obtained from a numerical solution to the problem of oscillation of a girder with N degrees of freedom. The numerical solution is obtained in the *Maple* system using the *Eigenvalues* operator for determining the eigenvalues of the matrix from the *LinearAlgebra* package of the *Maple* system.

The first frequency curves ω_1 obtained numerically and ω_D using formulas (4), (5) with coefficients (6), (7) are compared in graph 5 at $E = 2{,}1 \cdot 10^5\ MPa$, $F = 1{,}6 \cdot 10^{-3}\ м^2$, $m =$ 600$кг$, r = 5 m, h = 3 m and h = 4 m. As the number of panels increases, so does the accuracy of the analytical assessment. This is shown in graph 6 of the relative error $\varepsilon = (\omega_1 - \omega_D)/\omega_1$ on the number of panels.

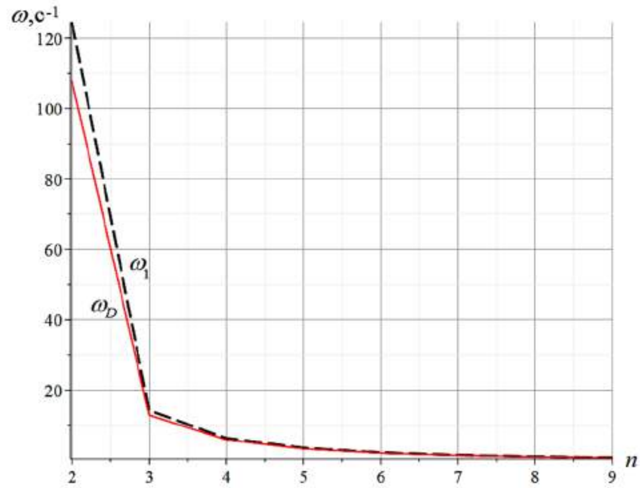

Fig. 5. Dependence on the number of panels first natural frequency ω_D of the spectrum calculated numerically at h = 3 m

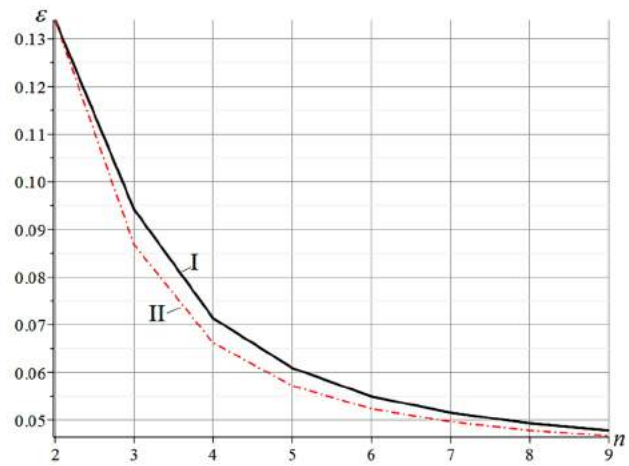

Fig. 6. Error dependence analytical estimate of the lower bound first frequency depending on the number of panels:
I — $H = 3$ m; II — $H = 4$ m

Conclusions. The scheme of a spatial statically definable regular tower-type structure is examined. An analytical solution has been found to the problem of structural deflections under the influence of various loads for an arbitrary number of panels. It is shown that the dependence of the horizontal displacement of the top on the radius of the tower has a minimum. For the corresponding curves, oblique asymptotes are found whose angle of inclination is calculated analytically.

The suggested design can be used in industrial and civil construction, and analytical solutions enable a simple preliminary calculation of the structure to be obtained at the design stage.

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